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Poincaré contraction of SU(1,1) Fock–Bargmann structure

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Abstract. Some aspects of the contraction process $\mathrm{SO}_0(1,2)$ to Poincaré are studied in this paper. The starting point is the choice of a suitable parametrization for the de Sitterian phase space $\mathrm{SO}_0(1,2)/\mathrm{SO}(2) \cong \mathrm{SU}(1,1)/\mathrm{U}(1)$. We show that the contraction to Poincaré must be realized by restricting the Fock–Bargmann space to a specific subspace. This constraint is necessary to make the divergent terms disappear. In particular, the classical result according to which the discrete series representation of $\mathrm{SU}(1,1)$ contracts onto the Wigner representation $P(m)$ is described at a global level.

1. Introduction

In the first approaches to (Galilean) kinematics, the best way of visualizing the motion of a body is to use spacetime diagrams. Later at a more abstract level, phase-space diagrams are introduced, mainly because they correspond to the set of all accessible (classical) states of a body.

As soon as the student has learnt a little group theory, he is able to consider geometrical objects as group cosets, namely

$$\text{Spacetime} = \text{Kinematical Group}/\text{Pure motions} \times \text{Rotations}$$

$$\text{Phase space} = \text{Kinematical Group}/\text{Time} \times \text{Rotations}$$

On the other hand the same student has learnt of the existence of several possible kinematics or relativities [1]. This means that different kinematical Lie groups are possible. For instance, for two-dimensional spacetime, we can distinguish between seven kinematics:

$$\begin{tikzcd} \mathrm{SO}_0(2,1) \arrow[r] \arrow[dr] & \text{Newton}_- \\ & \text{Poincaré} \arrow[r] & \text{Galileo} \arrow[r] & \text{Static} \\ \mathrm{SO}_0(1,2) \arrow[r] \arrow[ur] & \text{Newton}_+ \end{tikzcd}$$

Two of them are of 'maximal' symmetry, i.e. their kinematical groups are the 'de Sitterian' pseudo-orthogonal groups $\mathrm{SO}_0(1,2)$ and $\mathrm{SO}_0(2,1)$. The difference between them comes from the compact or non-compact nature of the time translation subgroup. Here, no physical unit is necessary to standardize their three (pseudo) angular parameters. They are the departure point for successive contractions (arrows on the diagram) until the ultimate 'kinematics' is reached where nothing moves. At each step, some of the parameters acquire a physical dimension. Hence, they may be interpreted as a length, time or momentum. Correspondingly, part of the simple group structure breaks down into a semi-direct product structure.

In quantum mechanics, the spacetime representation is favoured for historical reasons: in Galilean quantum mechanics the probabilistic interpretation of the wave-function and the description of the interaction in terms of potentials depending on the spacetime variables are strongly related to the non-ambiguous existence of a position operator [2]. Well known difficulties arise in Poincaré quantum mechanics for interacting systems: no satisfying position operator can be defined, localizability is in conflict with causality, inconsistencies arise in theories for high-spin interacting systems. They certainly become worse in de Sitter quantum mechanics. However in the case of $\mathrm{SO}_0(1,2)$ with compact time, the phase-space alternative exists in a natural and attractive way. The phase space $\mathrm{SO}_0(1,2)/\mathrm{SO}(2)$ is the simplest example of a classical domain and very rich analytic structures live on it [3]: Fock–Bargmann spaces of holomorphic functions that carry the discrete series representations of $\mathrm{SO}_0(1,2)$. It is remarkable to notice that Perelomov coherent states insure a straightforward quantization [4, 5] and that a theory of localization for quantum systems on phase space may be developed, using the coherent state formalism and the related existence of reproducing kernels [6].

In this work we examine several features of this 'analytic' quantum mechanics on the de Sitterian phase space $\mathrm{SO}_0(1,2)/\mathrm{SO}(2) \cong \mathrm{SU}(1,1)/\mathrm{U}(1)$ realized as the open unit disk $\mathcal{D}$. Actually we adopt the 'flat-limit' point of view by studying some aspects of the contraction process $\mathrm{SO}_0(1,2) \to$ Poincaré $P_+^\uparrow(1,1)$. We do not obtain a Poincaré phase-space quantum mechanics [7, 8] this way but rather momentum Wigner quantum mechanics. One interesting byproduct that results from this paper can be summarized as follows.

Let $h(z)$ be holomorphic in $\mathcal{D}$ such that

$$F(z,\bar{z}) = N\left(\frac{\hbar\kappa}{mc}\right)\left(\frac{1-|z|^2}{1+z^2}\right)^{mc/\hbar\kappa} h(z) \tag{1.1}$$

is square integrable on $\mathcal{D}$ with respect to the measure element

$$\mathrm{d}\mu(z,\bar{z}) = (1-|z|^2)^{-2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}. \tag{1.2}$$

Here, m is the Poincaré mass of the elementary system and κ is the de Sitterian curvature parameter. Then the function defined by

$$\Phi(p) = h\left(\frac{\mathrm{i}p}{p_0+mc}\right) \tag{1.3}$$

where $p_0 = (p^2+m^2c^2)^{1/2}$ is square integrable on $\mathbb{R}$ with respect to the invariant measure element $\mathrm{d}p/p_0$.

In our opinion the following important points should be investigated/clarified in the future.

(i) The behaviour of distributions on the Foch–Bargmann space should be studied in order to reach, by contraction, more objects in the Gelfand triplet for $L^2(\mathbb{R}, \mathrm{d}p/p_0)$.

(ii) The flat limit of eigensolutions to the de Sitterian Schrödinger-like equation on $\mathcal{D}$

$$HF = EF$$

where $F = F(z, \bar{z})$ is square integrable and vanishes at the boundary $\mathcal{D}$, and H has the form

$$H = L_0 + \varepsilon V(L_0, L_1, L_{01})$$

should be investigated. $L_0 \equiv z\partial_z - \bar{z}\partial_{\bar{z}} + (mc/\hbar\kappa)$ is the de Sitterian 'time-translation' generator, and V some symmetric operator-valued function of the three $\mathrm{SO}_0(1,2)$ generators L_0, L_1, L_{01}. The spectrum of L_0 is $\{(mc/\hbar\kappa) + n, n \in \mathbb{N}\}$ if we impose some polarization condition on F. More generally Lie algorithms and semiclassical methods [9, 10] could be systematically worked out to study the commutativity of 'multi' contractions on the various parameters $m, c, \hbar, \kappa$ and ϵ.

The organization of the paper is as follows. In section 2, we describe three factorizations of the double covering $\mathrm{SU}(1,1)$ of $\mathrm{SO}_0(1,2)$ that are physically relevant for the description of spacetime, phase space and de Sitterian spatial infinity or phase-space infinity respectively. The first one is non-standard whereas the two others are well known in semi-simple group theory [11]. The third section combines the two first factorizations in order to give the phase space $\mathcal{D}$ a '(q,p)' parametrization. The existence of a $\mathrm{U}(1)$ 'gauge freedom' allows one to consider different parametrizations labelled by an arbitrary function λ. Also we give an interesting interpretation of two particular parametrizations in terms of Lobatchevskian geometry. In section 4, we recall the Fock–Bargmann structure on $\mathcal{D}$ as being associated with quantum elementary systems for the kinematical group $\mathrm{SU}(1,1)$. The problem of the contraction onto Poincaré is approached. We insist here on the analytic aspect and give a 'semi-classical' expansion of the general element of the Fock–Bargmann space in terms of the curvature parameter. In section 5, the contraction of the unitary irreducible representation (UIR) at a global level is performed by restricting it to the adapted subspace of functions (1.1). This allows one to check once more the classical result according to which the discrete series representation of $\mathrm{SU}(1,1)$ contracts onto the Wigner positive-energy UIR $P(m)$ of the Poincaré group.

2. Relativistic meaning of SU(1,1)

$\mathrm{G} = \mathrm{SU}(1,1)$ is the group of 2×2 complex matrices g of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C} \tag{2.1}$$

with unit determinant $\det g = |\alpha|^2 - |\beta|^2 = 1$. It follows that $|\alpha| \geqslant 1$ for any $g \in \mathrm{SU}(1,1)$.

Three types of decomposition of G are relevant to our physical interpretation of $\mathrm{SU}(1,1)$. The first one is called the 'spacetime' factorization and is defined by the involution (where the superscript t denotes transposition)

$$i_j : g \longrightarrow g^{\mathrm{t}}. \tag{2.2}$$

Explicitly, any g of the form (2.1) can be written as

$$g = jl \qquad \text{with } ll^{\mathrm{t}} = e,\ e \text{ being the identity.} \tag{2.3}$$

The subgroup L of $\mathrm{SU}(1,1)$ determined by $l^{\mathrm{t}} = l^{-1}$ is isomorphic to $\mathrm{SO}(1,1)$:

$$\mathrm{L} = \left\{ l = \begin{pmatrix} \cosh(\phi/2) & \mathrm{i}\sinh(\phi/2) \\ -\mathrm{i}\sinh(\phi/2) & \cosh(\phi/2) \end{pmatrix} \varepsilon,\ \phi \in \mathbb{R} \right\}. \tag{2.4}$$

where $\varepsilon = \pm e$. The element j in (2.3) is non-uniquely determined by

$$\begin{aligned} jj^{\mathrm{t}} = gg^{\mathrm{t}} &= \begin{pmatrix} \alpha^2+\beta^2 & 2\,\mathrm{Re}\,\alpha\bar\beta \\ 2\,\mathrm{Re}\,\alpha\bar\beta & \bar\alpha^2+\bar\beta^2 \end{pmatrix} \\ &\equiv \begin{pmatrix} \cosh\psi\mathrm{e}^{\mathrm{i}\theta} & \sinh\psi \\ \sinh\psi & \cosh\psi\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \end{aligned} \tag{2.5}$$

with $0 \leqslant \theta < 2\pi$ and $\psi \in \mathbb{R}$. Namely

$$\theta = \arg(\alpha^2+\beta^2) \qquad \psi = \sinh^{-1}(\alpha\bar\beta + \bar\alpha\beta). \tag{2.6}$$

A possible solution to (2.5) is

$$j = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2} \end{pmatrix} \begin{pmatrix} \cosh(\psi/2) & \sinh(\psi/2) \\ \sinh(\psi/2) & \cosh(\psi/2) \end{pmatrix} \tag{2.7}$$

which leads to the (global but not unique) decomposition of $\mathrm{SU}(1,1)$

$$g = \varepsilon\exp(\theta N_0)\exp(\psi N_1)\exp(\phi N_{01}). \tag{2.8}$$

Here N_0, N_1, N_{01} denote the Lie algebra elements

$$N_0 = (\mathrm{i}/2)\sigma_3 \qquad N_1 = \tfrac{1}{2}\sigma_1 \qquad N_{01} = -\tfrac{1}{2}\sigma_2 \tag{2.9}$$

with the commutation rules

$$[N_0, N_1] = N_{01} \qquad [N_{01}, N_0] = N_1 \qquad [N_{01}, N_1] = N_0. \tag{2.10}$$

Given (2.6), ϕ is determined in (2.8) by $l = j^{-1}g\varepsilon$:

$$\tanh(\phi/2) = -\mathrm{i}\frac{\beta - \bar\alpha\tanh(\psi/2)\mathrm{e}^{\mathrm{i}\theta}}{\alpha - \bar\beta\tanh(\psi/2)\mathrm{e}^{\mathrm{i}\theta}}. \tag{2.11}$$

$(\theta,\psi) \equiv j$ defined by (2.5)–(2.7) is, in fact, a system of global coordinates for anti de Sitter spacetime. To see this, let us introduce the three coordinates in $\mathbb{R}^3$:

$$y^1 = \kappa^{-1}\sinh\psi \qquad y^2 = \kappa^{-1}\cosh\psi\cos\theta \qquad y^0 = \kappa^{-1}\cosh\psi\sin\theta \tag{2.12}$$

where the 'curvature' κ^{-1} is inverse length-like. Equivalently, we have

$$jj^{\mathrm{t}} = \begin{pmatrix} \kappa y_+ & \kappa y^1 \\ \kappa y^1 & \kappa y_- \end{pmatrix} \equiv \Gamma(y) \tag{2.13}$$

with $y_\pm = y^2 \pm \mathrm{i} y^0$. $\mathrm{SU}(1,1)$ acts on the $\Gamma(y)$ set and this action is induced from the left action of $\mathrm{SU}(1,1)$ on the set of matrices j, i.e.

$$g : j \longrightarrow j' : gj = j'l' \tag{2.14a}$$

and

$$\Gamma(y') = j'j'^{\mathrm{t}} = gjj^{\mathrm{t}}g^{\mathrm{t}} = g\Gamma(y)g^{\mathrm{t}}. \tag{2.14b}$$

The action is linear and determinant-preserving. Therefore the following pseudo-norm in $\mathbb{R}^3$ is left-invariant:

$$\det \Gamma(y) = \kappa^2((y^2)^2 + (y^0)^2 - (y^1)^2) = 1. \tag{2.15}$$

In particular, this sets up the isomorphism $\mathrm{SU}(1,1)/\mathbb{Z}_2 \cong \mathrm{SO}_0(2,1)$. The hyperboloid (2.15) is an embedding in $\mathbb{R}^3$ of the one-to-one anti de Sitter spacetime. We have seen that each point is in one-to-one correspondence with each class of the coset $SU(1,1)/SO(1,1)$. The *interpretation of the transformations generated* by N_0, N_1 and N_{01} is now clear. N_0 generates the 'translations in time' corresponding to $\mathrm{U}(1) \cong \mathrm{SO}(2)$, N_1 generates the 'translations in space' corresponding to one subgroup $\mathrm{SO}(1,1)$ and N_{01} generates the Lorentz transformations corresponding to the other $\mathrm{SO}(1,1)$. That the anti de Sitter spacetime is locally Minkowskian is also clear from the left-invariant metric $\mathrm{d}s^2$ in global coordinates

$$q^0 = \kappa^{-1}\theta \qquad q = \kappa^{-1}\psi \tag{2.16a}$$

given by

$$ds^2 = \cosh^2 \kappa q (dq^0)^2 - (dq)^2. \tag{2.16b}$$

The second decomposition of G is well known in semi-simple group theory [11]. It is called the Cartan decomposition but we shall call it the 'phase-space' decomposition for obvious reasons. It is defined by the Cartan involution

$$i_{ph} : g \longrightarrow (g\dagger)^{-1}. \tag{2.17}$$

The subgroup $\mathrm{H} = \mathrm{U}(1)$ is determined by $i_{ph}(g) = g$ whereas the condition $i_{ph}(g) = g^{-1}$ selects the subset P of Hermitian matrices in G. The decomposition $\mathrm{G} = \mathrm{PH}$ reads explicitly

$$g = p(z)h(\theta)$$

with

$$p(z) = \begin{pmatrix} \delta & \delta z \\ \delta \bar{z} & \delta \end{pmatrix} \qquad z = \beta\bar{\alpha}^{-1} \qquad \delta = (1 - |z|^2)^{-1/2} \tag{2.18}$$

and

$$h(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \qquad \theta = 2\arg\alpha \qquad 0 \leqslant \theta < 4\pi. \tag{2.19}$$

The bundle section $z \epsilon \mathcal{D} \longrightarrow p(z) \in \mathrm{P}$ gives the open unit disk $\mathcal{D} = \{z \in \mathbb{C},\ |z| < 1\}$ a symmetric space realization as the coset space G/H. We remark that $p^2 = gg\dagger$ and that $(p(z))^{-1} = p(-z)$. $\mathrm{SU}(1,1)$ acts on $\mathcal{D}$ by the left action on the set of matrices $p(z)$, i.e.

$$g : p(z) \longrightarrow p(z') \text{ defined by } gp(z) = p(z')h'. \tag{2.20}$$

Explicitly, the action of g on z is homographic

$$z' \equiv g \cdot z = (\alpha z + \beta)(\bar{\beta} z + \bar{\alpha})^{-1}. \tag{2.21}$$

As is well known [5, 13], $\mathcal{D}$ is a Kählerian manifold. It is a bounded version of the Poincaré upper half-plane or, equivalently, the simplest example of a Lobatchevski space. The Kählerian potential is given from the so-called Bergman kernel:

$$K(z, \bar{z}) = \pi^{-1}(1 - |z|^2)^{-2}. \tag{2.22}$$

This leads to the Riemannian metric

$$\begin{aligned} \mathrm{d}s_{ph}^2 &= 2\frac{\partial^2}{\partial z \partial \bar{z}} \ln K(z, \bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z} \\ &= 4(1 - |z|^2)^{-2}\, \mathrm{d}z\, \mathrm{d}\bar{z} \end{aligned} \tag{2.23}$$

and to the $\mathrm{SU}(1,1)$-invariant Kählerian 2-form

$$\begin{aligned} \omega &= \mathrm{i}\frac{\partial^2}{\partial z \partial \bar{z}} \ln K(z, \bar{z})\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 2\mathrm{i}(1 - |z|^2)^{-2}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned} \tag{2.24}$$

The third decomposition is also standard [11]. It is the Iwasawa decomposition $\mathrm{G} = \mathrm{HAN}$, with $\mathrm{H} = \mathrm{U}(1)$, $\mathrm{A} = \mathrm{SO}(1,1)$, $\mathrm{N} \cong \mathbb{R}$. Explicitly,

$$g = h(\theta)\varepsilon b(\psi) n(x) \tag{2.25}$$

with

$$h(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{2.26}$$

where

$$\theta = 2\arg(\alpha + \beta) \qquad \varepsilon = e \qquad \text{for } 0 \leqslant \arg(\alpha + \beta) < \pi$$

$$\theta = 2\arg(\alpha + \beta) - 2\pi \qquad \varepsilon = -e \qquad \text{for } \pi \leqslant \arg(\alpha + \beta) < 2\pi$$

$$b(\psi) = \begin{pmatrix} \cosh(\psi/2) & \sinh(\psi/2) \\ \sinh(\psi/2) & \cosh(\psi/2) \end{pmatrix} \qquad \psi = 2\ln|\alpha + \beta| \tag{2.27}$$

$$n(x) = \begin{pmatrix} 1 + \mathrm{i}x & -\mathrm{i}x \\ \mathrm{i}x & 1 - \mathrm{i}x \end{pmatrix} \qquad x = \frac{\mathrm{Im}\, \alpha\bar{\beta}}{|\alpha + \beta|^2}. \tag{2.28}$$

The map $z \in \mathrm{U}(1) \cong S^1 \longrightarrow h(\arg z) \in H$ gives to the (Shilov) boundary S^1 of $\mathcal{D}$ a homogeneous space realization as the coset space $\mathrm{G}/\mathbb{Z}_2\mathrm{AN} \cong \mathrm{H}/\mathbb{Z}_2$. The semi-direct product group $\mathbb{Z}_2\mathrm{AN} \cong \mathbb{Z}_2 \times (\mathrm{A} \otimes \mathrm{N})$ is isomorphic to the affine group of $\mathbb{R}$ (easily checked from $b(\psi)n(x)b(-\psi) = n(e^{\psi}x)$). $\mathrm{SU}(1,1)$ acts on the boundary S^1 and this action is induced from the left action on the set of matrices $h(\theta)$:

$$g : h(\theta) \longrightarrow h(\theta')$$

defined by

$$gh(\theta) = h(\theta')\varepsilon' b' n'. \tag{2.29}$$

Explicitly we have

$$\mathrm{e}^{\mathrm{i}\theta'} = (\alpha \mathrm{e}^{\mathrm{i}\theta} + \beta)(\bar{\beta}\mathrm{e}^{\mathrm{i}\theta} + \bar{\alpha})^{-1}. \tag{2.30}$$

We recover the action (2.21) on $\mathcal{D}$ extended to its boundary, or the action (2.14*b*) when we extend it to the set of matrices

$$\hat{\Gamma}(y) = \begin{pmatrix} y_+ & y^1 \\ y^1 & y_- \end{pmatrix} \qquad \det \hat{\Gamma}(y) = |y_+|^2 - (y^1)^2 = 0 \tag{2.31}$$

with $\mathrm{e}^{\mathrm{i}\theta} = y^1/y_-$. This limit point on S^1 is obtained from the ratio

$$y^1/y_- = \mathrm{e}^{\mathrm{i}\theta} \tanh \psi$$

determined from (2.12) by letting ψ go to infinity. Since we have the relation (2.16*a*) for ψ, this means that either we let q go to infinity (κ being fixed) or we let κ go to infinity (q being fixed). In the first case S^1 will be seen as the anti de Sitter spatial infinity and in the second one as the projective null cone in $\mathbb{R}^3$.

3. A configuration–momentum parametrization of the phase space $\mathcal{D}$

Any matrix representation (2.18) of a point z in $\mathcal{D}$ admits a spacetime factorization (2.3). Explicitly, we have

$$p(z) = h(\theta)b(\psi)l(\phi) = j(\theta, \psi)l(\phi) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{3.1}$$

with

$$\begin{aligned} \alpha &= \varepsilon \mathrm{e}^{\mathrm{i}\theta/2}(\cosh(\psi/2)\cosh(\phi/2) - \mathrm{i}\sinh(\psi/2)\sinh(\phi/2)) \\ \beta &= \varepsilon \mathrm{e}^{\mathrm{i}\theta/2}(\sinh(\psi/2)\cosh(\phi/2) + \mathrm{i}\cosh(\psi/2)\sinh(\phi/2)). \end{aligned} \tag{3.2}$$

From the definition (2.18) of $p(z)$, we must have $\delta = \alpha$ real. This corresponds to the following constraint on θ:

$$\tan(\theta/2) = \tanh(\psi/2)\tanh(\phi/2) \tag{3.3a}$$

which implies

$$e^{i\theta} = \frac{\cosh\psi + \cosh\phi + i\sinh\psi\sinh\phi}{1 + \cosh\psi\cosh\phi}. \tag{3.3b}$$

Now we can explicitly write

$$z = \beta\bar{\alpha}^{-1} = e^{i\theta}\frac{\cosh\phi\sinh\psi + i\sinh\phi}{1 + \cosh\phi\cosh\psi} \tag{3.4a}$$

which, with (3.3), takes the final expression

$$z = z(\psi, \phi) = \frac{\sinh\psi + i\cosh\psi\sinh\phi}{1 + \cosh\phi\cosh\psi}. \tag{3.4b}$$

In order to give these expressions a more familiar meaning let us adopt a 'Minkowski–Lorentz' parametrization, namely (2.16*a*) together with

$$\sinh\phi = (p/mc) \qquad \cosh\phi = (p_0/mc) \tag{3.5a}$$

so that ϕ is given a 'rapidity' meaning. The vector (p_0, p) belongs to the forward mass hyperbola

$$\mathcal{V}_m^+ = \{(p_0, p) \in \mathbb{R}^2 \mid p_0 > 0, p_0^2 - p^2 = m^2c^2\}. \tag{3.5b}$$

First relation (3.3) imposes a specific value for q_0 which satisfies

$$e^{i\kappa q_0} = \frac{p_0 + mc\cosh\kappa q + ip\sinh\kappa q}{mc + p_0\cosh\kappa q}. \tag{3.6}$$

Second we deduce the coordinate transformation as the map

$$z \in \mathcal{D} \longleftrightarrow (q, p) \in \mathbb{R}^2$$

defined by

$$z(q, p, \kappa) = \frac{mc\sinh\kappa q + ip\cosh\kappa q}{mc + p_0\cosh\kappa q} \tag{3.7}$$

and conversely

$$\begin{aligned} q &= q(z, \bar{z}) = \frac{1}{\kappa}\sinh^{-1}\frac{z + \bar{z}}{1 - |z|^2} \\ p &= p(z, \bar{z}) = \frac{imc(\bar{z} - z)}{|1 + z^2|}. \end{aligned} \tag{3.8}$$

Note that the particular point $z = 0 \in \mathcal{D}$ is applied on $(q, p) = (0, 0) \in \mathbb{R}^2$. Moreover the map is evidently onto and

$$\lim_{(q,p)\to(\infty,\infty)} z(q, p, \kappa) = i \in S^1. \tag{3.9}$$

Let us also give the transformation of the derivatives. For example, we have

$$\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{mc}[(p_0 + mc\cosh\kappa q) - \mathrm{i}p\sinh\kappa q]\left[\frac{1}{\kappa}\partial_q - \frac{\mathrm{i}p_0}{\cosh\kappa q}\partial_p\right] \tag{3.10}$$

where $\partial_q = \partial/\partial q$ and $\partial_p = \partial/\partial p$. Finally, the Kählerian metric (2.23) and 2-form (2.24) are given in terms of (q,p) by

$$\mathrm{d}s^2 = \kappa^2\,\mathrm{d}q^2 + \cosh^2\kappa q(\mathrm{d}p/p_0)^2 \tag{3.11}$$

$$\omega = \kappa\cosh\kappa q\,\mathrm{d}q \wedge (\mathrm{d}p/p_0) = \mathrm{d}(\sinh\kappa q)\wedge(\mathrm{d}p/p_0). \tag{3.12}$$

The constraint (3.3) is specific of our representative choice (2.18) for the symmetric space $\mathcal{D}$. We could as well have chosen the section

$$p_\lambda(z_\lambda) = p(z_\lambda)k(\lambda) = \begin{pmatrix} \delta\mathrm{e}^{\mathrm{i}\lambda/2} & \delta z_\lambda \mathrm{e}^{-\mathrm{i}\lambda/2} \\ \delta\bar{z}_\lambda \mathrm{e}^{\mathrm{i}\lambda/2} & \delta\mathrm{e}^{-\mathrm{i}\lambda/2}\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \tag{3.13}$$

with α and β given by (3.2). The new constraint follows again from the reality of $\delta = \alpha\mathrm{e}^{-\mathrm{i}\lambda/2}$. We find with the preceding 'Minkowski-Lorentz' parametrization

$$\mathrm{e}^{\mathrm{i}(\kappa q_0 - \lambda)} = \frac{p_0 + mc\cosh\kappa q + \mathrm{i}p\sinh\kappa q}{mc + p_0\cosh\kappa q}. \tag{3.14}$$

It is then easy to show that z_λ only differs from z by a phase factor, i.e.

$$z_\lambda = z_\lambda(q,p,\kappa) = \mathrm{e}^{\mathrm{i}\lambda(z,\bar{z},\kappa)}z(q,p,\kappa). \tag{3.15}$$

This gauge freedom allows one to select different sections in the set of triplets (q_0,q,p) to define the (q,p) phase space. They are given by the expression of $q_0 = q_0(q,p)$ following from relation (3.14).

It is interesting to make a comment about the effect of such a parametrization. The metric (2.23) is transformed into

$$\begin{aligned} \mathrm{d}s_\lambda^2 &\equiv 4(1-|z_\lambda|^2)^{-2}\,\mathrm{d}z_\lambda\,\mathrm{d}\bar{z}_\lambda \\ &= 4(1-|z|^2)^{-2}(\mathrm{d}z\,\mathrm{d}\bar{z} + |z|^2\,\mathrm{d}\lambda^2 + \mathrm{i}(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z)\,\mathrm{d}\lambda) \end{aligned} \tag{3.16a}$$

while the 2-form (2.24) becomes

$$\omega_\lambda = [1 + \mathrm{i}(z\partial_z - \bar{z}\partial_{\bar{z}})\lambda]\omega. \tag{3.16b}$$

We first remark that this last expression is invariant with respect to the phase-space transformation

$$\lambda \longrightarrow \lambda + \chi(|z|). \tag{3.17}$$

Second there exists a specific function $\lambda = \lambda_1$ for which ω_λ is canonical. Indeed if we take

$$\lambda_1 = \tan^{-1}\left[\frac{\mathrm{i}(z^2 - \bar{z}^2)}{z^2 + \bar{z}^2 + 2}\right] \tag{3.18}$$

we find

$$\omega_1 = \frac{(1-|z|^2)(1+|z|^2)}{|(1+z^2)|^2}\omega = \frac{2\mathrm{i}(1+|z|^2)}{|(1+z^2)|^2(1-|z|^2)}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \tag{3.19}$$

or in (q,p) coordinates

$$\omega_1 = \frac{\kappa}{mc}\,\mathrm{d}q\wedge\mathrm{d}p. \tag{3.20}$$

The corresponding metric (3.16*a*) may be written as

$$\mathrm{d}s_\lambda^2 \equiv \mathrm{d}s_1^2 = \kappa^2\frac{p_0^2}{m^2c^2}\,\mathrm{d}q^2 + \frac{\mathrm{d}p^2}{p_0^2}. \tag{3.21}$$

It is remarkable to notice that this choice corresponds to taking $q_0 = 0$ in (3.14).

The definition of a time (q_0) in terms of space (q) and momentum (p) in such a way that ω becomes the familiar canonical 2-form $\omega_1 \equiv$ (3.20) is reminiscent of what we usually do in Galilean classical mechanics [14]. It is shown elsewhere that $(\kappa q, p/mc)$ has really a position–velocity meaning at the time $q_0 = 0$ for a test particle of mass m in anti de Sitter spacetime [8]. Otherwise $(\kappa q, p/mc)$ coordinatizes the set of classical free motions. The corresponding parametrization for the phase space $\mathcal{D}$ reads

$$z_1(q,p,\kappa) = \frac{p_0\sinh\kappa q + \mathrm{i}p}{p_0\cosh\kappa q + mc}. \tag{3.21}$$

We now come to the geometrical interpretation of the parametrization (3.4) and the constraint (section choice) (3.3). Our understanding takes place within the Lobatchevskian geometry framework. For instance, the coordinate curves corresponding to (3.4) are, at ϕ constant, circular arcs (the so-called constant-rapidity Lobatcheskian lines)

$$|z - \mathrm{i}\coth\phi|^2 = \frac{1}{\sinh^2\phi} \qquad |z| \leqslant 1 \tag{3.22}$$

orthogonal to the unit circle S^1. On the other hand, they are, at ψ constant, the circular arcs (the so-called constant-position Lobatcheskian lines)

$$\left|z + \frac{1}{\sinh\psi}\right|^2 = \coth^2\psi \qquad |z| \leqslant 1 \tag{3.23}$$

that join the two poles $z = \pm\mathrm{i}$. The two curves are evidently orthogonal at $z = z(\phi,\psi)$. Note that the points at infinity for the constant-rapidity Lobatcheskian lines are given by

$$z_{\pm\infty} = \pm\frac{1}{\cosh\phi} + \mathrm{i}\tanh\phi. \tag{3.24}$$

The meaning of the angle θ in equation (3.3) is now clear. The de Sitterian rotation angle θ defining the Cartan decomposition (3.1) is also the polar angle of the tangent to the constant-rapidity Lobatcheskian lines at $z = z(\phi,\psi)$.

The geometrical construction of the coordinate curves corresponding to the 'Galilean' parametrization $z_1 = (3.21)$ is clear from the relationships

$$z_1 = \mathrm{e}^{-\mathrm{i}\theta} z(\psi, \phi) = \mathrm{i}\bar{z}(\phi, \psi). \tag{3.25}$$

There is a permutation of the constant-rapidity lines and constant-position lines together with an axis permutation. The constant-rapidity lines are now circular arcs joining the points $z = \pm 1$ and tangent to the constant-position lines for z, and vice versa.

Note that in the Lobatcheskian terminology [15] the constant-position lines for z are hypercycles equidistant to the vertical Lobatcheskian line defined by $\mathrm{Re}\, z = 0$ and $|z| \leqslant 1$. The Lobatcheskian motions (2.21), when restricted to the Lorentz subgroup (2.4), translate points along a given hypercycle while leaving fixed the 'absolute' points $z = \pm\mathrm{i}$.

All these comments are illustrated in figure 1.

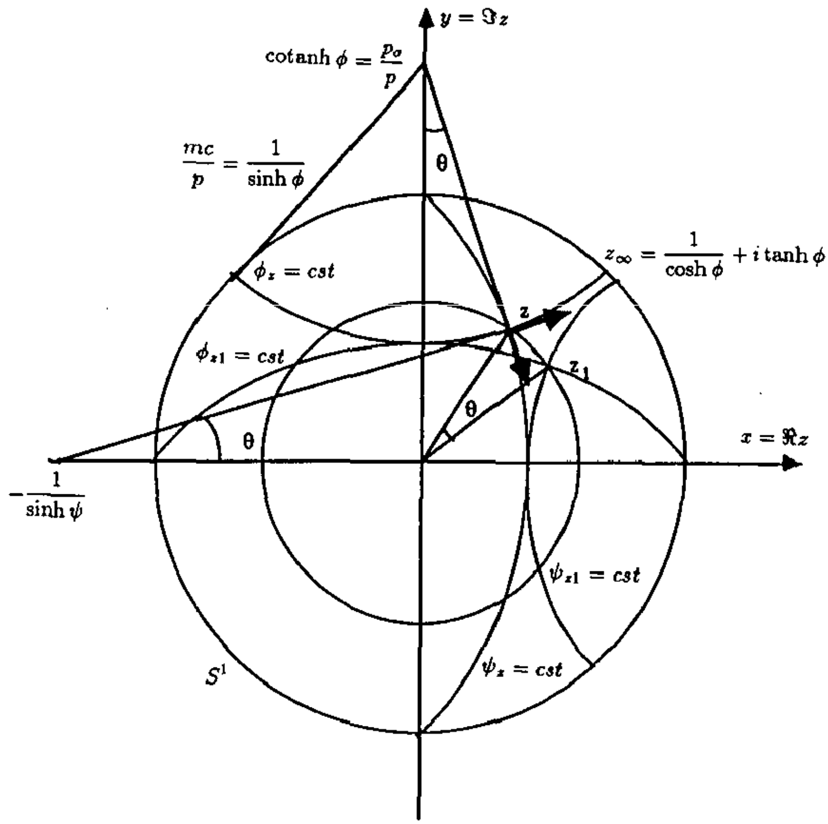

Figure 1. Anti de Sitterian phase space and Lobatchevskian geometry of the disk. Coordinate curves corresponding to $z = (3.4)$ are, at ϕ constant, the circular arcs (3.22) orthogonal to S^1. They are, at ϕ constant, the circular arcs (3.23) that join the two poles $z = \pm\mathrm{i}$. The Galilean parametrization $z_1 = \mathrm{e}^{-\mathrm{i}\theta} z$ corresponds to a permutation of the previous circular arcs together with an (xy) permutation.

4. Fock–Bargmann structures for SU(1,1)

Geometric quantization of the classical phase space $\mathcal{D}$ leads to a Hilbert space structure which is appropriated to anti de Sitter quantum mechanics. The open unit disk $\mathcal{D} \cong \mathrm{SU}(1,1)/\mathrm{U}(1)$ can be viewed as an orbit for the coadjoint representation of $\mathrm{SU}(1,1)$. This picture allows one to build up the UIR of $\mathrm{SU}(1,1)$ which can be associated with elementary quantum systems for anti de Sitter relativity in agreement with Wigner's formulation of symmetry in quantum mechanics. All these mathematical procedures together with the physical interpretation is described elsewhere [8]. We just recall here the main results which are necessary to our purpose.

We denote $\mathcal{F}^{E_0} = \{f(z) : z \in \mathcal{D}\}$ the Fock–Bargmann space [5] of funtions holomorphic inside the unit circle, satisfying the square-integrability condition $(f, f) < \infty$ with respect to the scalar product

$$(f_1, f_2) = \frac{E_0 - \frac{1}{2}}{2\pi} \int_{\mathcal{D}} \bar{f}_1(z) f_2(z)(1 - |z|^2)^{2E_0} \omega \tag{4.1}$$

with ω given by (2.24) and $E_0 > \frac{1}{2}$ a fixed real number. The representation operator $T^{E_0}(g)$ of $\mathrm{SU}(1,1)$ is defined by

$$(T^{E_0}(g) f)(z) = (\beta z + \bar{\alpha})^{-2E_0} f(g^t \cdot z). \tag{4.2}$$

T^{E_0} is unitary, irreducible and belongs to the discrete series for $E_0 > \frac{1}{2}$ [11, 12] (actually we should restrict to integers or half-integers larger than or equal to 1 in order to stick to the *strictu-senso* definition of discrete series for $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2, \mathbb{R})$, but the distinction has no importance if we deal with the Lie algebra $\mathrm{SU}(1,1)$ or with the universal covering of $\mathrm{SU}(1,1)$, as was noticed by Barut and Fronsdal [12]).

Since our final aim is a better understanding of some aspects of the contraction process $\mathrm{SU}(1,1) \longrightarrow \mathcal{P}_+^\uparrow(1,1)$, as the curvature κ goes to zero and E_0 goes to the infinity, it is apparent that the form (4.1) or equivalently the space $\mathcal{F}^{E_0}$ is not well adapted to such a limit procedure. Therefore we introduce a 'weighted' Fock–Bargmann space

$$\mathcal{F}_W^{E_0} = \{F(z, \bar{z}) = (1 - |z|^2)^{E_0} f(z),\ f \in \mathcal{F}^{E_0}\} \tag{4.3}$$

which is the Hilbert space of square-integrable non-analytic functions inside the unit disk with the scalar product

$$(F_1, F_2) = \frac{E_0 - \frac{1}{2}}{2\pi} \int_{\mathcal{D}} \bar{F}_1(z, \bar{z}) F_2(z, \bar{z}) \omega \tag{4.4}$$

and with the 'polarization' constraint issued from the analyticity of f:

$$(\partial_{\bar{z}} + E_0 z(1 - |z|^2)^{-1}) F(z, \bar{z}) = 0. \tag{4.5}$$

The representation operator $T_W^{E_0}(g)$ on $\mathcal{F}_W^{E_0}$ is deduced from $T_0^E(g)$ defined by (4.2).

$$(T_W^{E_0}(g) F)(z, \bar{z}) = \left(\frac{\bar{\beta}\bar{z} + \alpha}{\beta z + \bar{\alpha}}\right)^{E_0} F(g^t \cdot z, g^t \cdot \bar{z}). \tag{4.6}$$

The corresponding action of the Lie algebra $\mathrm{su}(1,1)$ is obtained as usual by differentiation of (4.6). From the decomposition (2.8) of $g \in \mathrm{SU}(1,1)$, we easily get

$$N_0 \longrightarrow L_0 = z\partial_z - \bar{z}\partial_{\bar{z}} + E_0 \tag{4.7a}$$

$$N_1 \longrightarrow L_1 = \frac{\mathrm{i}}{2}((1-z^2)\partial_z + (1-\bar{z}^2)\partial_{\bar{z}} + E_0(\bar{z}-z)) \tag{4.7b}$$

$$N_{01} \longrightarrow L_{01} = -\tfrac{1}{2}((1+z^2)\partial_z - (1+\bar{z}^2)\partial_{\bar{z}} + E_0(\bar{z}+z)) \tag{4.7c}$$

with the commutation rules

$$[L_0, L_1] = \mathrm{i}L_{01} \qquad [L_{01}, L_0] = \mathrm{i}L_1 \qquad [L_{01}, L_1] = \mathrm{i}L_0. \tag{4.8}$$

Let us mention that these operators (4.7) commute with the polarization operator in (4.5).

If we adopt another choice of parametrization for the domain $\mathcal{D}$, or more precisely the one resulting from the gauge transformation (3.15), the corresponding definition of the space $\mathcal{F}^{E_0}_{W,\lambda}$ will be given by

$$\mathcal{F}^{E_0}_{W,\lambda} = \{F_\lambda(z_\lambda, \bar{z}_\lambda) = (1-|z_\lambda|^2)^{E_0}\mathrm{e}^{\mathrm{i}E_0\lambda} f(z_\lambda),\ f \in \mathcal{F}^{E_0}\}. \tag{4.9}$$

The polarization condition (4.5) and the group actions (4.6) and (4.7) have to be modified as a consequence. However in the sequel we shall stick to the original 'purely Cartan' definition (4.3), with appropriate comments about the more general situation (see section 5).

The last step before the contraction procedure is to translate this abstract machinery where no operational physical quantity appears into the familiar language where physical dimensions are present. Besides the (three) fundamental constants κ, m and c already injected into the formalism, the quantum context now introduces action-dimensional physical quantitites at the order of $\hbar$. The unique dimensionless combination of these four constants is the parameter

$$\xi = \frac{\hbar\kappa}{mc} \tag{4.10}$$

which is typical of a de Sitter quantum mechanics. The pure number E_0, which is actually the minimal weight of the representation $T^{E_0}_W$, is a meromorphic function of ξ with a simple pole at $\xi = 0$, i.e.

$$E_0 = E_0(\xi) = \xi^{-1} + \sum_{n\geqslant 0} e_n \xi^n \tag{4.11}$$

where $e_0, e_1, \ldots,$ are pure numbers (maybe equal to zero). Let us also introduce the dimensionless quantities

$$u = mcq/\hbar \qquad v = p/mc \qquad v_0 = (1+v^2)^{1/2} \tag{4.12}$$

so that it is easy to show that the parametrization of $\mathcal{D}$ in terms of $z =$ (3.7) now reads

$$z = z(u,v,\xi) = \frac{\sinh\xi u + \mathrm{i}v\cosh\xi u}{1 + v_0\cosh\xi u} \tag{4.13a}$$

and we have

$$1-|z|^2=2(1+v_0\cosh\xi u)^{-1}. \tag{4.13b}$$

Actually we have to say more on the behaviour of an arbitrary state $F(z,\bar z)\in\mathcal{F}_W^{E_0}$ in terms of ξ. Let us rename this function in order to take into account its dependence on ξ, u and v:

$$F\equiv F(z(u,v,\xi),\bar z(u,v,\xi),\xi)=\exp\{E_0(\xi)\ln(1-|z|^2)\}f(z(u,v,\xi),\xi) \tag{4.14a}$$

where a possible explicit dependence on ξ has also been introduced in $f\in\mathcal{F}^{E_0}$. Using (4.13*b*), it reads

$$F(z(u,v,\xi),\bar z(u,v,\xi),\xi)=\exp\left\{-E_0(\xi)\ln\left(\frac{1+v_0\cosh\xi u}{2}\right)\right\}f(z(u,v,\xi)). \tag{4.14b}$$

Because of the form of (4.14*b*) it is convenient to introduce the logarithms of this function, i.e. to define

$$\begin{aligned}G(z(u,v,\xi),\bar z(u,v,\xi))&\equiv\ln F(z,\bar z,\xi)\\&=-E_0(\xi)\ln\tfrac12(1+v_0\cosh\xi u)+g(z(u,v,\xi))\end{aligned} \tag{4.15}$$

where

$$g(z(u,v,\xi),\xi)\equiv\ln f(z). \tag{4.16}$$

Now the polarization constraint on G issued from (4.5) reads

$$\partial_{\bar z}G=-E_0z(1-|z|^2)^{-1}. \tag{4.17}$$

Introducing the new quantities (4.12) in the derivation operator (3.13) we easily find

$$\partial_{\bar z}\equiv\frac{\partial}{\partial\bar z}=\frac12(v_0+\cosh\xi u+\mathrm{i}v\sinh\xi u)\left(\xi^{-1}\partial_u+\frac{\mathrm{i}v_0}{\cosh\xi u}\partial_v\right). \tag{4.18}$$

Thus equation (4.17) becomes

$$\begin{aligned}&(v_0+\cosh\xi u+\mathrm{i}v\sinh\xi u)\left(\xi^{-1}\partial_u+\frac{\mathrm{i}v_0}{\cosh\xi u}\partial_v\right)G\\&\qquad+E_0(\xi)(\sinh\xi u+\mathrm{i}v\cosh\xi u)=0.\end{aligned} \tag{4.19}$$

Due to expression (4.11) for $E_0(\xi)$, the term $E_0\ln\frac12(1+v_0\cosh\xi u)$ in $G\equiv(4.15)$ has a simple pole at $\xi=0$. Thus let us expand $g\equiv(4.16)$ accordingly to finally write

$$G=-E_0(\xi)\ln\tfrac12(1+v_0\cosh\xi u)+\xi^{-1}\sum_{n=0}^{\infty}\xi^n g_{n-1}(u,v). \tag{4.20}$$

From equation (4.19) satisfied by $G \equiv$ (4.20) follows an infinite sequence of correlated partial differential equations for the unknown functions $g_{n-1}(u,v)$. Because our aim is finally to take the limit for $\xi \to 0$, it is sufficient to give the first terms of the expansion. They are explicitly computed in appendix A.

We then conclude on the behaviour of $F \equiv$ (4.14) by taking the exponential of G. We obtain

$$F(\xi,u,v) = \exp -(\xi^{-1}F_{-1}(u,v) + F_0(u,v) + \xi F_1(u,v) + \mathrm{O}(\xi^k)) \qquad k \geqslant 2 \tag{4.21}$$

$$F_{-1}(u,v) = \ln \tfrac{1}{2}(1+v_0) - \varphi_{-1}(v) \tag{4.22a}$$

$$F_0(u,v) = \mathrm{i}uv_0\varphi'_{-1}(v) + e_0 \ln \tfrac{1}{2}(1+v_0) - \varphi_0(v) \tag{4.22b}$$

$$\begin{aligned} F_1(u,v) = {} & \frac{u^2}{2}\left(\frac{v_0}{1+v_0} + v_0\varphi'_{-1}(v) + (1+v^2)\varphi''_{-1}(v)\right) \\ & + \mathrm{i}uv_0\varphi'_0(v) + e_1 \ln \tfrac{1}{2}(1+v_0) - \varphi_1(v) \end{aligned} \tag{4.22c}$$

where φ_{-1}, φ_0 and φ_1 are arbitrary functions of v obtained in solving g_{-1}, g_0 and g_1 (cf appendix A) and φ', φ'' are the first and second derivatives with respect to v. Such an analysis shows clearly that the asymptotic behaviour is ruled by the singular term $\exp -\xi^{-1}F_{-1}(u,v)$. It will disappear if $F_{-1}(u,v) = 0$ i.e. if we have

$$\varphi_{-1}(v) = \ln \tfrac{1}{2}(1+v_0)). \tag{4.23}$$

This particular case will be extensively discussed in the following section. The Gaussian which is present in (4.22*c*) at the order ξ of (4.21) (let us recall that from (4.12) u is associated with the position q) is characteristic of the free de Sitterian states where an 'universal' harmonic oscillator strength ξ is present [1].

5. Contraction of a Fock–Bargmann structure

It is well known [7] that the representation $T^{E_0}(g)$ or $T^{E_0}_W(g)$ of $\mathrm{SU}(1,1) \simeq \mathrm{SO}_0(2,1) \times \mathbb{Z}_2$ must contract to the Wigner positive-energy representation $P(m)$ of $P^{\uparrow}_{+}(1,1)$ as κ (or ξ) goes to zero and E_0 goes to infinity while keeping the product κE_0 equal to $mc/\hbar$ (this justifies the first term in the expansion (4.10) of $E_0(\xi)$).

Let us see how it works explicitly on the space $\mathcal{F}^{E_0}_W$. In order to eliminate the singular terms in the expansion (4.21) of $F(z,\bar{z})$, we must impose some constraints on the form of $F \equiv$ (4.3) and more particularly of the original analytic function $f(z)$. Indeed it is easy to show that the function $(1+z^2)^{-1}$ analytic in $\mathcal{D}$ has a limit

$$\lim_{\xi \to 0}(1+z^2)^{-1} = \tfrac{1}{2}(1+v_0). \tag{5.1}$$

Then to cancel $F_{-1} \equiv$ (4.22*a*) we must factorize $f(z)$ as

$$f(z,\xi) = N(\xi)(1+z^2)^{-E_0(\xi)}h(z,\xi) \tag{5.2}$$

where the function h is now analytic in both $z \in \mathcal{D}$ and $\xi \geqslant 0$. $N(\xi)$ is a normalization factor possibly non-analytic in ξ. In the following normalization will not be imposed in order to ignore this non-analytic $N(\xi)$. The square-integrability condition now reads

$$\|h\|^2_{\mathrm{reg}} = \frac{E_0 - \frac{1}{2}}{2\pi} \int_{\mathcal{D}} |h(z,\xi)|^2 \left(\frac{1-|z|^2}{|1+z^2|}\right)^{2E_0} \omega < \infty. \tag{5.3}$$

Note that the weight regular factor is such that

$$\lim_{\xi \to 0} \left(\frac{1-|z|^2}{|1+z^2|}\right)^{2E_0(\xi)} = 1. \tag{5.4}$$

We accordingly restrict our considerations by working on the subspace of $\mathcal{F}^{E_0}_W$ which consists of functions of the form

$$F(z,\bar{z},\xi) = \left(\frac{1-|z|^2}{1+z^2}\right)^{E_0(\xi)} h(z) \tag{5.5}$$

where $h(z) \equiv h(z,0)$. Note that this choice fixes the arbitrary functions $\varphi_{-1}(v)$, $\varphi_0(v)$, and $\varphi_1(v)$ in (4.22) (cf appendix A for details). The expansion (4.21) then reads

$$F(u,v,\xi) = \left(\exp\left(-\mathrm{i}\frac{uv}{1+v_0}\right) h\bigg|_{z=\mathrm{i}v/(1+v_0)}\right) \exp[-\xi F_1 + \mathrm{O}(\xi^k)] \tag{5.6}$$

where

$$F_1(u,v,\xi) = \frac{u^2}{2} + \frac{u}{1+v_0}\left(\mathrm{i}ve_0 - \frac{h'}{h}\bigg|_{z=\mathrm{i}v/(1+v_0)}\right). \tag{5.7}$$

We now introduce

$$\Psi(q,p) \equiv \lim_{\xi \to 0} F(u,v,\xi) = \exp(-\mathrm{i}(\boldsymbol{q}_s \cdot \boldsymbol{p})/\hbar)\Phi(p) \tag{5.8}$$

where

$$\Phi(p) = h(\mathrm{i}p/(p_0 + mc)). \tag{5.9}$$

The 2-vectors $\boldsymbol{p}$ and $\boldsymbol{q}_s$ are given by

$$\boldsymbol{p} = (p_0, p) \qquad \boldsymbol{q}_s = \left(q_0 = \frac{qp}{p_0 + mc}, q\right) \tag{5.10}$$

and

$$\boldsymbol{q}_s \cdot \boldsymbol{p} = q_0 p_0 - qp. \tag{5.11}$$

We recognize in (5.10) the flat spacetime limit of section 3.3 which defines, in the space of parameters (θ,ψ,ϕ) for $\mathrm{SU}(1,1)$, the Cartan phase space for this group. The remarkable properties of the de Sitterian section (5.10) for Poincaré are listed elsewhere [7].

Let us state the crucial result about the limit of the square-integrability condition:

Proposition. Let $h(z)$ be a holomorphic function in the unit disk $\mathcal{D}$ such that the integral

$$I_{E_0} = \frac{1}{4}\int_{\mathcal{D}} \left(\frac{1-|z|^2}{|1+z^2|}\right)^{2E_0} |h(z)|^2\omega \tag{5.12a}$$

is finite for all $E_0 \geqslant 1$. Then the function on $\mathbb{R}$ defined by (5.9) is square integrable on $\mathbb{R}$ with respect to the Lorentz invariant $\mathrm{d}p/p_0$.

Proof. (See [16] for more detailed results.) Let us first notice that (5.12*a*) can be written

$$I_{E_0} = \tfrac{1}{4}\int_{\mathcal{D}} (1-|z|^2)^{2E_0}|f(z)|^2\omega$$

with f as in (5.2) ($N(\xi)=1$). By putting $E_0 = 1$, this hypothesis implies

$$I_1 = \int_{\mathcal{D}} |f(z)|^2 \,\mathrm{d}\,\mathrm{Re}\, z\, \mathrm{d}\,\mathrm{Im}\, z < \infty.$$

It follows from the holomorphy of f: $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Indeed,

$$\begin{aligned} I_1 &= \lim_{r\to 1_-} \int_{|z|<r} \left(\sum_{m,n\geqslant 0} a_n \bar{a}_m z^n \bar{z}^m\right) \mathrm{d}\,\mathrm{Re}\, z\, \mathrm{d}\,\mathrm{Im}\, z \\ &= \frac{\pi}{2}\lim_{r\to 1_-} \sum_{n\geqslant 0} |a_n|^2 \frac{r^{2n+2}}{2n+2} = \frac{\pi}{2}\sum_{n\geqslant 0}\frac{|a_n|^2}{2n+2} \\ &\simeq \sum_{n\geqslant 0} \frac{|a_n|^2}{n} < \infty. \end{aligned} \tag{5.12b}$$

The last inequality results from the monotonic convergence theorem. On the other hand the function $\Phi(p)$ from (5.9) will be square integrable if the following is true:

$$J = \int_0^{\infty} \left| h\left(\frac{\mathrm{i}p}{p_0 + mc}\right)\right|^2 \frac{\mathrm{d}p}{p_0} < \infty.$$

After putting $x = p/(p_0 + mc)$ and replacing h by f, this integral becomes

$$J = 2\int_0^1 (1-x^2)|f(\mathrm{i}x)|^2 \,\mathrm{d}x < \infty.$$

Holomorphy allows one to expand in series under the integral and to use uniform convergence:

$$\begin{aligned} J &= 2\lim_{r\to 1_-} \sum_{m,n\geqslant 0} a_m \bar{a}_n (-1)^n i^{m-n} \int_0^r (x^{m+n} - x^{m+n+2})\,\mathrm{d}x \\ &\leqslant 2\sum_{m,n\geqslant 0} \frac{|a_m||a_n|}{(m+n+1)(m+n+3)} = 2\sum_{p\geqslant 1} \frac{1}{p(p+2)}\left(\sum_{k=1}^{p} |a_k|\,|a_{p-k}|\right) \\ &\leqslant 2\sum_{p\geqslant 1}\frac{1}{p^2}\left(\sum_{k=1}^{p}|a_k|^2\right) = 2\sum_{p\geqslant 1}|a_p|^2\left(\sum_{k=p}^{\infty}\frac{1}{k^2}\right). \end{aligned} \tag{5.13a}$$

Now it is easy to show that

$$\sum_{k=p}^{\infty} \frac{1}{k^2} \leqslant \frac{2}{p}.$$

It follows from (5.12b), (5.13a) and (5.13b) that J is finite. □

Let us now show how the representation $T_W^{\bar{E}_0}(g)$ of $\mathrm{SU}(1,1)$ on $F \equiv$ (5.6) contracts.

The first step is to write $g \in \mathrm{SU}(1,1)$ in the form

$$g \equiv \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = h(\kappa a_0) b(\kappa a) l(\phi) \tag{5.14}$$

where ϕ is here given by

$$\phi = \tanh^{-1}(k/k_0) \qquad k \in V_m^+. \tag{5.15}$$

Note that α and β are then explicitly given by (3.2) where $\theta = \kappa a_0$ and $\psi = \kappa a$. It is well known from the Inönü–Wigner contraction procedure [17] that, when $\kappa \to 0$, the set (a_0, a, k) turns out to be the set of parameters characterizing the Poincaré group

$$P_+^\uparrow(1,1) = \left\{ ((a_0, a), \Lambda_k) = \begin{pmatrix} 1 & 00 \\ a_0 & \Lambda_k \\ a & \end{pmatrix} \ \Lambda_k = \frac{1}{mc} \begin{pmatrix} k_0 & k \\ k & k_0 \end{pmatrix} \right\} \tag{5.16}$$

(see [7] for details).

The second step is to express the dependence on ξ (or κ) of (4.6) where F has the form (5.6). This will be done by replacing z by (4.13a) and α, β by their expression in terms of a_0, a and ϕ according to (5.14). The last step is to expand (4.6) in ξ and to take the limit when $\xi \to 0$. All the details concerning these two steps are given in appendix B.

Let us give the final result expressed in terms of the phase-space variables. We introduce the transformed variables (q', p') related to (u', v') given in appendix B (formulae (B5) and (B6)).

$$q' = q - \left(a_0 \frac{p}{mc} - a \frac{p_0}{mc} \right) \tag{5.17a}$$

and

$$p' = \frac{k_0}{mc} p - \frac{k}{mc} p_0 \qquad p_0' = \frac{k_0}{mc} p_0 - \frac{k}{mc} p \tag{5.17b}$$

so that we get

$$\begin{aligned} \lim_{\xi \to 0} (T_W^{E_0(\xi)}(g(a_0, a, k; \xi)) F)(z(q,p,\xi), \bar{z}(q,p,\xi)) &\equiv (U_m(a_0, a, k)\Psi)(q,p) \\ &= \mathrm{e}^{\mathrm{i}(\zeta_0 + \zeta_1)/\hbar} \Psi(q', p') \end{aligned} \tag{5.18}$$

where Ψ has been already defined in (5.11) and

$$\zeta_0 = \zeta_0(q,p;k) = \frac{-2m^2c^2qk}{((p_0+mc)(k_0+mc)-kp)} \tag{5.19a}$$

and

$$\zeta_1 = \zeta_1(p;a_0,a,k) = m^2c^2\frac{a_0(k_0+p_0)-a(k+p)}{m^2c^2+k_0p_0-kp}. \tag{5.19b}$$

Such a result requires some comments. First let us recall from [7] that the left coset space $\mathrm{SU}(1,1)/\mathrm{U}(1)$ characterized by $z \in \mathcal{D}$ has been contracted to the left coset space $P_+^\uparrow(1,1)/\mathrm{T}$, where T is the subgroup of time translations with the section [7] $\sigma : P_+^\uparrow(1,1)/\mathrm{T} \longrightarrow P_+^\uparrow(1,1)$

$$\sigma(q,p) = \left(\left(q_0 = \frac{qp}{p_0+mc}, q\right), \Lambda_p\right). \tag{5.20}$$

Moreover the action of $\mathrm{SU}(1,1)$ on z, i.e. $z \to g^{\mathrm{t}} \cdot z$ has been shown (in appendix B) to contract to the action of $P_+^\uparrow(1,1)$ on (q,p) explicitly given by (5.17*a*) and (5.17*b*). In fact such a transformation law is directly obtained from the action of $P_+^\uparrow(1,1)$ on the space $P_+^\uparrow(1,1)/\mathrm{T}$ given by

$$(q',p') = (\Lambda_k^{-1}(a_0,a), \Lambda_k^{-1})(q,p). \tag{5.21}$$

Next, the representation (5.18) gives the action of $P_+^\uparrow(1,1)$ on the function $\Psi(q,p)$. We can easily deduce the corresponding action on $\Phi(p)$ from its relation (5.11) to $\Psi(q,p)$. It turns out to be, as expected, the momentum version of the Wigner representation $P(m)$, i.e.

$$(U_m(a_0,a,k)\Phi)(p) = \exp(\mathrm{i}(a\cdot p)/\hbar)\Phi(p'). \tag{5.22}$$

Finally, the representation of the Poincaré algebra is obtained as usual by giving the infinitesimal generators P_0, P, and K [7]. If we consider the representation on the functions $\Psi(q,p)$ given by (5.18), we get

$$P_0\Psi(q,p) \equiv -\mathrm{i}\hbar\frac{\partial}{\partial a_0}U\Psi|_e = \left(\mathrm{i}\hbar\frac{p}{mc}\partial_q + mc\right)\Psi(q,p) \tag{5.23a}$$

$$P\Psi(q,p) \equiv -\mathrm{i}\hbar\frac{\partial}{\partial a}U\Psi|_e = \left(\mathrm{i}\hbar\frac{p_0}{mc}\partial_q + \frac{mcp}{p_0+mc}\right)\Psi(q,p) \tag{5.23b}$$

$$K\Psi(q,p) \equiv -\mathrm{i}mc\frac{\partial}{\partial k}U\Psi|_e = \left(\mathrm{i}p_0\partial_p - \frac{m^2c^2q}{\hbar(p_0+mc)}\right)\Psi(q,p) \tag{5.23c}$$

where all the derivatives are taken at $e \equiv (0,0,\mathbb{I})$ the identity of the group. Clearly, these generators satisfy the commutation relations

$$[P_0,P] = 0 \qquad [P_0,K] = -\mathrm{i}P \qquad [P,K] = -\mathrm{i}P_0. \tag{5.24}$$

As we go from representation (5.18) to (5.22), we immediately go from (5.23) to the well-known momentum Wigner representation on $\Phi(p)$, i.e.

$$P_0\Phi(p) = p_0\Phi(p) \qquad P\Phi(p) = p\Phi(p)$$
$$K\Phi(p) = \mathrm{i}p_0\frac{\mathrm{d}}{\mathrm{d}p}\Phi(p). \tag{5.25}$$

Finally, had we chosen the more general parametrization (3.15) of $\mathcal{D}$ defining the space $\mathcal{F}^{E_0}_{W,\lambda} \equiv$ (4.9), all the previous results would be modified in the following sense. The 'regular' functions (5.5) are now defined by

$$\begin{aligned} F_\lambda(z_\lambda, \bar{z}_\lambda, \xi) &\equiv \mathrm{e}^{\mathrm{i}E_0(\xi)\lambda(z,\bar{z},\xi)} F(z_\lambda, \bar{z}_\lambda, \xi) \\ &= \left(\frac{1-|z_\lambda|^2}{1+z_\lambda^2}\right)^{E_0(\xi)} \mathrm{e}^{\mathrm{i}E_0(\xi)\lambda(z,\bar{z},\xi)} h(z_\lambda). \end{aligned} \tag{5.26}$$

In terms of the phase-space variables (q, p) we have

$$\lim_{\xi\to 0} F_\lambda = \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{mcqp}{p_0+mc} - \frac{\hbar}{mc}p_0\partial_\xi\lambda|_0\right)\right\}\Psi(p) \tag{5.27}$$

where $\lambda = \lambda(z(u,v,\xi), \bar{z}(u,v,\xi), \xi)$ is such that

$$\lim_{\xi\to 0}\lambda = 0 \qquad \lim_{\xi\to 0}\partial_\xi\lambda = \partial_\xi\lambda|_0. \tag{5.28}$$

The expression (5.27) may be again written

$$\Psi_\lambda(q,p) \equiv \lim_{\xi\to 0} F_\lambda = \exp(-\mathrm{i}(q_{s,\lambda}\cdot p)/\hbar)\Phi(p) \tag{5.29}$$

with

$$q_{s,\lambda} = \left(q_0 = \frac{qp}{p_0+mc} + \frac{\hbar}{mc}\partial_\xi\lambda|_0, q\right). \tag{5.30}$$

Note that (5.30) corresponds to the flat spacetime limit of section (3.14). As a final comment it is easy to see that the 'Galilean' choice $\lambda = \lambda_1 =$ (3.18) can be written

$$\lambda_1 = \tan^{-1}\left(-\frac{p\sinh\kappa q}{p_0 + mc\cosh\kappa q}\right) \tag{5.31a}$$

so that

$$\frac{\hbar}{mc}\partial_\xi\lambda_1|_0 = \partial_\kappa\lambda_1|_0 = \frac{-qp}{p_0+mc} \tag{5.31b}$$

and this leads to the expected choice of section $q_{s,\lambda_1} = (0, q)$.

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Appendix A

The solution to equation (4.17) takes the form (4.20). By expanding (4.17) in ξ as ξ tends to zero, we will be able to determine the unknown functions $g_{n-1}(u,v)$. Indeed the function

$$g(z(u,v,\xi),\xi) \equiv \xi^{-1} \sum_{n=0}^{\infty} \xi^n g_{n-1}(u,v) \tag{A1}$$

clearly satisfies the homogeneous equation (corresponding to (4.17)

$$(v_0 + \cosh \xi u + \mathrm{i} v \sinh \xi u)\left(\xi^{-1}\partial_u + \frac{\mathrm{i} v_0}{\cosh \xi u}\partial_v\right) g(z,\xi) = 0. \tag{A2}$$

Inserting (A1) into (A2), we get (for $\xi \neq 0$)

$$\mathrm{i} v_0 \sum_{n=0}^{\infty} \xi^{n+1} \partial_v g_{n-1}(u,v) + \sum_{n,k=0}^{\infty} \xi^{n+2k} \frac{u^{2k}}{(2k)!} \partial_u g_{n-1}(u,v) = 0. \tag{A3}$$

Assume that it is true for all ξ, we get an infinite sequence of partial differential equations, i.e.

$$\partial_u g_{-1}(u,v) = 0 \tag{A4a}$$

and

$$\mathrm{i} v_0 \partial_v g_{j-2}(u,v) + \sum_{k=0}^{[(j-1)/2]} \frac{u^{2k}}{(2k)!} \partial_u g_{j-2k-1}(u,v) = 0 \qquad j \geqslant 1. \tag{A4b}$$

Equation (A4a) corresponds to the coefficient which is independent of ξ while equation (A4b) corresponds to the coefficient of ξ^j. Equation (A4a) is solved directly as

$$g_{-1}(u,v) = \varphi_{-1}(v). \tag{A5}$$

The following functions, $g_l(u,v)$, $l \geqslant 0$, are obtained by recursively solving equation (A4b) and using (A5). For example, g_0 is the solution of

$$\partial_u g_0 + \mathrm{i} v_0 \partial_v g_{-1} = 0$$

i.e.

$$g_0(u,v) = -\mathrm{i} u v_0 \varphi'_{-1}(v) + \varphi_0(v) \tag{A6}$$

where $\varphi'_{-1}(v) \equiv \mathrm{d}\varphi_{-1}(v)/\mathrm{d}v$. The function g_1 must satisfy

$$\partial_u g_1 + \mathrm{i} v_0 \partial_v g_0 = 0$$

so that the solution is

$$g_1(u,v) = \frac{-u^2}{2}(v\varphi'_{-1}(v) + (1+v^2)\varphi''_{-1}(v)) - \mathrm{i}uv_0\varphi'_0(v) + \varphi_1(v). \tag{A7}$$

The other functions may be determined in the same way but we are not interested in them here. The expansion of $g \equiv$ (A1) then leads to

$$\begin{aligned} g(z(\xi,u,v),\xi) &= \xi^{-1}\varphi_{-1}(v) + (\varphi_0(v) - \mathrm{i}uv_0\varphi'_{-1}(v)) \\ &\quad + \xi\left(\varphi_1(v) - \mathrm{i}uv_0\varphi'_0(v) - \frac{u^2}{2}(v\varphi'_{-1}(v) + (1+v^2)\varphi''_{-1}(v))\right) \\ &\quad + \mathrm{O}(\xi^k) \qquad k \geqslant 2. \end{aligned} \tag{A8}$$

Now the expansion of G $\equiv$ (4.19) is easily obtained as (using $E_0(\xi) = $ (4.11))

$$\begin{aligned} \mathrm{G} &= \xi^{-1}(\varphi_{-1}(v) - \ln \tfrac{1}{2}(1+v_0)) \\ &\quad + \varphi_0(v) - e_0 \ln \tfrac{1}{2}(1+v_0) - \mathrm{i}uv_0\varphi'_{-1}(v) \\ &\quad + \xi\Bigg(\varphi_1(v) - e_1 \ln \tfrac{1}{2}(1+v_0) + \mathrm{i}uv_0\varphi'_0(v) \\ &\quad + \frac{u^2}{2}\left(\frac{v_0}{1+v_0} + v\varphi'_{-1}(v) + (1+v^2)\varphi''_{-1}(v)\right)\Bigg) + \mathrm{O}(\xi^k). \end{aligned} \tag{A9}$$

For the particular choice (5.2) of f which would delete the singularity in ξ, we get $g \equiv$ (A1) of the form

$$g(z(\xi,u,v),\xi) = -E_0(\xi)\ln(1+z^2) + \ln h(z). \tag{A10}$$

Expanding the two members of (A10) in ξ we have

$$\begin{aligned} &\xi^{-1}\sum_{n=0}^{\infty} \xi^n g_{n-1}(u,v) \\ &\quad = \xi^{-1}\ln\tfrac{1}{2}(1+v_0) + \ln h|_{z=\mathrm{i}v/(1+v_0)} + e_0 \ln\tfrac{1}{2}(1+v_0) - \frac{\mathrm{i}uv}{1+v_0} \\ &\quad + \xi\Bigg(e_1 \ln \tfrac{1}{2}(1+v_0) - \frac{u}{1+v_0}\left(\mathrm{i}v_0e_0 - \frac{h'}{h}\bigg|_{z=\mathrm{i}v/(1+v_0)}\right) \\ &\quad - \frac{u^2}{2(1+v_0)}\Bigg) + \mathrm{O}(\xi^k). \end{aligned} \tag{A11}$$

Comparing with (A8), we find

$$\begin{aligned} &\varphi_{-1}(v) = \ln\tfrac{1}{2}(1+v_0), \varphi_1(v) = e_1 \ln \tfrac{1}{2}(1+v_0) \\ &\varphi_0(v) = e_0 \ln \tfrac{1}{2}(1+v_0) + \ln h|_{z=\mathrm{i}v/(1+v_0)} \end{aligned} \tag{A12}$$

and G $\equiv$ (A9) becomes

$$\begin{aligned} \mathrm{G} &= -\frac{\mathrm{i}uv}{1+v_0} + \ln h|_{z=\mathrm{i}v/(1+v_0)} \\ &\quad - \xi\left(\frac{u^2}{2} + \frac{u}{1+v_0}\mathrm{i}ve_0 - \frac{h'}{h}\bigg|_{z=\mathrm{i}v/(1+v_0)}\right) + \mathrm{O}(\xi^k) \qquad k \geqslant 2. \end{aligned} \tag{A13}$$

Appendix B

Let us show how the representation $T_W^{E_0}(g)$ of $\mathrm{SU}(1,1)$ contracts. It is given by (4.6) with $F = (5.6)$. Here again it is more convenient to introduce the logarithmic function

$$G^g(z,\bar z) = \ln(T_W^{E_0}(g)F)(z,\bar z). \tag{B1}$$

The dependence on ξ is introduced through $z = (4.13a)$ and the expression of α and β as given from (5.14), i.e. $(\kappa = (mc/\hbar)\xi)$

$$\begin{aligned}\alpha &= \pm e^{i\kappa a_0/2}\left(\cosh\frac{\kappa a}{2}\cosh\frac{\phi}{2} - i\sinh\frac{\kappa a}{2}\sinh\frac{\phi}{2}\right)\\ \beta &= \pm e^{i\kappa a_0/2}\left(\sinh\frac{\kappa a}{2}\cosh\frac{\phi}{2} + i\cosh\frac{\kappa a}{2}\sinh\frac{\phi}{2}\right).\end{aligned} \tag{B2}$$

Let us notice that from $z = (4.13a)$ we find

$$\lim_{\xi\to 0} z = \frac{iv}{1+v_0} \qquad \lim_{\xi\to 0}\partial_\xi z = \frac{u}{1+v_0}. \tag{B3}$$

Moreover the expression of $z' = (2.21)$ in terms of ξ may be given by using (B2) and (4.13a) but we will omit this here. Instead we will write

$$\lim_{\xi\to 0} z' = \frac{iv'}{1+v_0'} \qquad \lim_{\xi\to 0}\partial_\xi z' = \frac{u'}{1+v_0'} \tag{B4}$$

where the transformed quantities v', v_0' and u' are respectively given by

$$v' = \frac{k_0}{mc}v - \frac{k}{mc}v_0 \qquad v_0' = \frac{k_0}{mc}v_0 - \frac{k}{mc}v \tag{B5}$$

and

$$u' = u + \frac{mc}{\hbar}(-a_0 v + a v_0). \tag{B6}$$

This is the transformation law of (u,v) under $P_+^\uparrow(1,1)$.

Let us now write $G^g \equiv$ (B1) by using (4.6) and (5.6):

$$G^g(z,\bar z) = E_0\ln\left(\frac{\bar\beta\bar z + \alpha}{\beta z + \bar\alpha}\right) + G(z',\bar z'). \tag{B7}$$

$G(z',\bar z')$ has exactly the expansion (A13) but at the transformed point (u',v'), i.e.

$$G(z',\bar z') = -\frac{iu'v'}{1+v_0'} + \ln h|_{z=iv'/(1+v_0')} + O(\xi^k) \qquad k \geqslant 1. \tag{B8}$$

The expansion in ξ of the other term of (B7) must be computed. In fact, we have

$$\lim_{\xi\to 0}\left(\frac{\bar\beta\bar z + \alpha}{\beta z + \bar\alpha}\right) = 1 \tag{B9}$$

while

$$\lim_{\xi\to 0}\partial_\xi\left(\frac{\bar\beta\bar z+\alpha}{\beta z+\bar\alpha}\right)=-\frac{\mathrm{i}u(k+mc(v-v'))}{mc(1+v_0)(1+v_0')}+\frac{\mathrm{i}(a_0(k_0+mcv_0)-a(k+mcv))}{\hbar(1+v_0')}. \tag{B10}$$

This last expression can be simplified by using the relation

$$(1+v_0)(1+v_0')=\frac{((1+v_0)(k_0+mc)-vk)^2}{mc(k_0+mc)}. \tag{B11}$$

We finally obtain

$$\lim_{\xi\to 0}\partial_\xi\left(\frac{\bar\beta\bar z+\alpha}{\beta z+\bar\alpha}\right)=-\frac{2\mathrm{i}uk}{(1+v_0)(k_0+mc)-vk}+\frac{\mathrm{i}mc}{\hbar}\frac{a_0(k_0+mcv_0)-a(k+mcv)}{mc+k_0v_0-kv} \tag{B12}$$

and

$$E_0(\xi)\ln\left(\frac{\bar\beta\bar z+\alpha}{\beta z+\bar\alpha}\right)=\lim_{\xi\to 0}\partial_\xi\left(\frac{\bar\beta\bar z+\alpha}{\beta z+\bar\alpha}\right)+\mathrm{O}(\xi^k)\qquad k\geqslant 1. \tag{B13}$$

Combining (B8) and (B13) we get the expansion in ξ of $G^g\equiv$ (B7), i.e.

$$G^g=\mathrm{i}(\zeta_0(u,v;k)+\zeta_1(v;a_0,a,k))-\mathrm{i}\frac{u'v'}{1+v_0'}+\ln h|_{z=\mathrm{i}v'/(1+v_0')}+\mathrm{O}(\xi^k) \tag{B14}$$

with

$$\zeta_0(u,v;k)=\frac{-2uk}{(1+v_0)(k_0+mc)-vk} \tag{B15}$$

and

$$\zeta_1(v;a_0,a,k)=\frac{mc}{\hbar}\frac{a_0(k_0+mcv_0)-a(k+mcv)}{mc+k_0v_0-kv}. \tag{B16}$$

Back to the original F and the representation $T_W^{E_0}$ we find

$$\lim_{\xi\to 0}T_W^{E_0(\xi)}(g(a_0,a,k;\xi)F)(z(u,v,\xi),\bar z(u,v,\xi))=\mathrm{e}^{\mathrm{i}(\zeta_0(u,v;k)+\zeta_1(v;a_0,a,k))}\mathrm{e}^{-\mathrm{i}u'v'/(1+v_0')}h|_{z=\mathrm{i}v'/(1+v_0')}. \tag{B17}$$

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